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1994 J. Phys. A: Math. Gen. 27 4919

(http://iopscience.iop.org/0305-4470/27/14/019)

J. Phys. A: Math. Gen. 27 (1994) 4919–4924. Printed in the UK

A new twist in the kinematics and elastic dynamics of thin filaments and ribbons

I Klapper and M Tabor
Program in Applied Mathematics, University of Arizona, Tucson, AZ 85721, USA

Received 24 February 1994

Abstract. A formula governing the evolution of twist in moving filaments or ribbons of finite extent is derived. This evolution is shown to be made up of a 'dynamic' part corresponding to physical properties of the filament or ribbon and a 'geometric' part due to the motion of the filament or ribbon core itself. These results are used to extend classical elastic rod theory to the case of motion including dynamically evolving twist. In addition, the averaged geometric contribution is noted to be minus the time rate of change of the writhing number and it is shown that the writhe is a conserved quantity for closed filaments moving according to certain integrable curve dynamics.

The occurrence of thin twisted filamentary structures is commonplace in many problems of physics. We present here a theory for the dynamics of such structures under imposed motions or internal and external forces. The possible applications of such a theory include, among other things, the dynamics of proteins and supercoiled DNA [1], writhing instability in fibres and cables [2], motions of vortex tubes [3], magnetic flux tubes and modelling of sunspot formation [4]. This is not a new subject; motions of infinitesimally thin filaments (without twist) have been considered under a number of guises [3, 5], and the static theory of thin elastic rods is classical [6]. Here we consider the *dynamics of twisted ribbons and filaments of finite but small width* in which the twist of a finite-width filament results in forces that drive the dynamics of the core filament.

We begin with the definition of a twisted filament. Consider a time-dependent differentiable space curve $\boldsymbol{X}(\sigma, t)$ parametrized by a material (or intrinsic) parameter $0 \leqslant \sigma \leqslant L$, i.e. one that moves with the curve, and let s be arclength (often t or σ dependence will be taken to be implicit). Define $A(\sigma, t) \equiv \mathrm{d}s/\mathrm{d}\sigma$. We denote spatial derivatives with respect to σ using a dash and t derivatives using a dot. The tangent vector $\boldsymbol{T}$ of the curve $\boldsymbol{X}$ is given by $\boldsymbol{T} = A^{-1}\boldsymbol{X}'$. As $\boldsymbol{T}$ is a unit vector then $\boldsymbol{T}\cdot(\mathrm{d}\boldsymbol{T}/\mathrm{d}s) = 0$ and, following standard conventions, we set $\mathrm{d}\boldsymbol{T}/\mathrm{d}s = A\boldsymbol{T}' \equiv \kappa\boldsymbol{N}$ where the curvature $\kappa(\sigma, t)$ is defined by $\kappa \equiv |\mathrm{d}\boldsymbol{T}/\mathrm{d}s|$. *The unit vector* $\boldsymbol{N}(\sigma, t)$ *is called the normal vector*; wherever $\kappa \neq 0$, $\boldsymbol{N}$ is well defined and we can form a local coordinate triad by defining the binormal vector $\boldsymbol{B}(\sigma, t)$ to be $\boldsymbol{B} \equiv \boldsymbol{T} \times \boldsymbol{N}$. For simplicity we assume that $\kappa = 0$ occurs only at an isolated set of points.

Twist is introduced in the following manner: let $\boldsymbol{X}$ be the centreline of a filament with some slight thickness ϵ. For simplicity we will assume that the cross section of the filament is always circular with constant radius ϵ. Straighten out the filament, cutting if necessary (i.e. if $\boldsymbol{X}$ is closed). Given a constant unit vector field $\boldsymbol{V}$ on $\boldsymbol{X}$ with $\boldsymbol{T}\cdot\boldsymbol{V} = 0$, we define a reference ribbon $(\boldsymbol{X}, \boldsymbol{X} + \epsilon\boldsymbol{V})$ to be the ribbon consisting of the points $\boldsymbol{X}(\sigma) + \alpha\boldsymbol{V}$, $0 \leqslant \sigma \leqslant L$, $0 \leqslant \alpha \leqslant \epsilon$. (The construction of a reference ribbon is fairly arbitrary; however,

0305-4470/94/144919+06$19.50

this method of definition will be convenient later when elastic forces are introduced.) Now deform and twist the filament into the desired configuration, reconnecting ends if necessary. For simplicity we assume that $\boldsymbol{T}(\sigma,t)\cdot\boldsymbol{V}(\sigma,t)=0$ remains true for all t and σ. The reference ribbon may be considered to be half a longitudinal cross section of the filament, and the twist of the filament is the same as the twist of the reference ribbon. The twist $\omega(\sigma,t)$ of the filament can be calculated using the reference ribbon. Since $\boldsymbol{V}(\sigma,t)$ is a unit vector, $(\mathrm{d}/\mathrm{d}s)\boldsymbol{V}=\boldsymbol{\Lambda}\times\boldsymbol{V}$ for some $\boldsymbol{\Lambda}(\sigma,t)$. The twist is then given by

$$\omega(\sigma,t)=\boldsymbol{\Lambda}(\sigma,t)\cdot\boldsymbol{T}(\sigma,t)=\left(\boldsymbol{V}\times\frac{\mathrm{d}}{\mathrm{d}s}\boldsymbol{V}\right)\cdot\boldsymbol{T} \tag{1}$$

i.e. the rate that $\boldsymbol{V}$ twists around the central curve $\boldsymbol{X}$ measured with respect to arclength. In the same manner (1) also defines the twist of a ribbon. We note that $A\omega=(\boldsymbol{V}\times\boldsymbol{V}')\cdot\boldsymbol{T}$ is the twist measured with respect to the material coordinate σ.

We first consider kinematics. We allow the filament (or ribbon) to move continuously with a prescribed velocity of the central curve $\boldsymbol{X}(\sigma,t)$ plus a prescribed twisting around that curve, requiring only that the cross section of the filament remain perpendicular to $\boldsymbol{X}$ (i.e. no longitudinal shear); in particular $\boldsymbol{T}(\sigma,t)\cdot\boldsymbol{V}(\sigma,t)=0$. We consider the effects of these two types of motion separately. First hold $\boldsymbol{X}$ steady and allow twisting. We then have $\dot{\boldsymbol{V}}=f(\sigma,t)\boldsymbol{T}\times\boldsymbol{V}$ for some given $f(\sigma,t)$. Now, over an element of the filament from $\boldsymbol{X}(\sigma)$ to $\boldsymbol{X}(\sigma+\Delta\sigma)$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\sigma}^{\sigma+\Delta\sigma}A\omega\,\mathrm{d}\sigma&=\frac{\mathrm{d}}{\mathrm{d}t}\Delta\theta\\&=f(\sigma+\Delta\sigma,t)-f(\sigma,t)\end{aligned}$$

where $\theta=\int^{s(\sigma)}\omega\,\mathrm{d}s=\int^{\sigma}A\omega\,\mathrm{d}\sigma$ is the angle of rotation of the reference ribbon at $\boldsymbol{X}(\sigma)$. Thus when $\boldsymbol{X}$ is fixed,

$$\frac{\mathrm{d}}{\mathrm{d}t}(A\omega)=f'. \tag{2}$$

This is a twist conservation law; $(\mathrm{d}/\mathrm{d}t)A\omega\Delta\sigma$, the rate of change of total twist in an element of the filament, is equal to the flux of twist through the ends.

Now translate the filament without twisting (this is analogous to parallel transport). Then $\dot{\boldsymbol{V}}=\boldsymbol{\Omega}\times\boldsymbol{V}$ for some $\boldsymbol{\Omega}$ with $\boldsymbol{\Omega}\cdot\boldsymbol{T}=0$. As $\boldsymbol{T}\cdot\dot{\boldsymbol{T}}=0$ we can write $\boldsymbol{\Omega}=\beta\dot{\boldsymbol{T}}+\gamma\boldsymbol{T}\times\dot{\boldsymbol{T}}$ for some β and γ. Using the condition $\boldsymbol{V}\cdot\boldsymbol{T}=0$, an easy calculation shows that $\beta=0$ and $\gamma=1$. *Now although filament twisting has been ruled out for the moment, it is still possible for ω to change due to motion of $\boldsymbol{X}$ itself*. We calculate this change as follows: again consider a short element of the filament between $\boldsymbol{X}(\sigma)$ and $\boldsymbol{X}(\sigma+\Delta\sigma)$. We can find $(\mathrm{d}/\mathrm{d}t)\Delta\theta$, the change in angle of rotation over the material element, by calculating the rotation rate (engendered by the motion of $\boldsymbol{X}$) of $\boldsymbol{V}(\sigma+\Delta\sigma)$ around $\boldsymbol{T}(\sigma)$. (Remember that $\boldsymbol{V}(\sigma)$ does not rotate relative to $\boldsymbol{T}(\sigma)$.) We find

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\sigma}^{\sigma+\Delta\sigma}A\omega\,\mathrm{d}\sigma&=\frac{\mathrm{d}}{\mathrm{d}t}\Delta\theta\\&=\boldsymbol{\Omega}(\sigma+\Delta\sigma))\cdot\boldsymbol{T}(\sigma)\\&=A\kappa(\sigma)[\boldsymbol{B}(\sigma)\cdot\dot{\boldsymbol{T}}(\sigma)]\Delta\sigma\end{aligned}$$

using the relations $\boldsymbol{\Omega}=\boldsymbol{T}\times\dot{\boldsymbol{T}}$, $\boldsymbol{T}'=A\kappa\boldsymbol{N}$, and $\boldsymbol{B}=\boldsymbol{T}\times\boldsymbol{N}$. In the limit $\Delta\sigma\to 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t}(A\omega)=A\kappa(\dot{\boldsymbol{T}})_B=A\kappa\left(\frac{\mathrm{d}}{\mathrm{d}s}\dot{\boldsymbol{X}}\right)_B \tag{3}$$

where $(\cdot)_B$ stands for the binormal component.

Write $\dot{\boldsymbol{X}} = q_T\boldsymbol{T} + q_N\boldsymbol{N} + q_B\boldsymbol{B}$ (wherever $\kappa \neq 0$). Using the Frenet–Serret equations [7]

$$\frac{\mathrm{d}}{\mathrm{d}s}\boldsymbol{T} = \kappa\boldsymbol{N} \qquad \frac{\mathrm{d}}{\mathrm{d}s}\boldsymbol{N} = -\kappa\boldsymbol{T} + \tau\boldsymbol{B} \qquad \frac{\mathrm{d}}{\mathrm{d}s}\boldsymbol{B} = -\tau\boldsymbol{N} \tag{4}$$

(the scalar τ is the torsion) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}(A\omega) = \kappa(q_B' + A\tau q_N). \tag{5}$$

From this point forward we assume that $\kappa \neq 0$ except possibly at an isolated set of points. There is no mathematical or physical necessity to this assumption; we make it merely because the Frenet triad $\boldsymbol{T}$, $\boldsymbol{N}$, $\boldsymbol{B}$, is a conveniently (and commonly) used basis. In general, one could choose a basis arbitrarily, in which case the form of equations (4) (and thus subsequent equations) would be altered. Now, putting (2) and (5) together gives

$$\frac{\mathrm{d}}{\mathrm{d}t}(A\omega) = f' + \kappa(q_B' + A\tau q_N) \tag{6}$$

for general motions of the filament or ribbon. We point out an analogy between the change in ω under motions of a ribbon and the geometric phases found in classical and quantum mechanics [8]. Under this analogy the first term on the right of (6) arises from the motion of $\boldsymbol{X} + \epsilon\boldsymbol{V}$ around $\boldsymbol{X}$ and can be considered to be the 'dynamical' variation of ω, while the second term arises from the transport of the ribbon by motions of $\boldsymbol{X}$ and can be considered as the 'geometrical' variation of ω. We note, however, that the latter variation has physical relevance without reference to the closed paths that are usual in the theory of geometrical phases. Furthermore when we come to the consideration of elastic forces, both parts of (6) will have direct energetic consequences. We also note that there are some interesting parallels between the ribbon evolution formula (3) and recent considerations of anholonomic phases in magnetic chains [9].

Now, dotting $\boldsymbol{X}' = A\boldsymbol{T}$ with $\boldsymbol{T}$ and taking a time derivative gives

$$\dot{A} = A\left(\frac{\mathrm{d}}{\mathrm{d}s}\dot{\boldsymbol{X}}\right)_{\boldsymbol{T}} = q_T' - A\kappa q_N. \tag{7}$$

Equations (6) and (7) together with the known evolution equations (see Keener [3])

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(A\kappa) &= \left[\frac{q_N'}{A} + \kappa q_T - \tau q_B\right]' - \tau(q_B' + A\tau q_N) \\
\frac{\mathrm{d}}{\mathrm{d}t}(A\tau) &= \left[\frac{1}{A\kappa}\left(\frac{q_B'}{A} + \tau q_N\right)' + \frac{\tau}{\kappa}\left(\frac{q_N'}{A} - \tau q_B\right) + \tau q_T\right]' + \kappa(q_B' + A\tau q_N)
\end{aligned} \tag{8}$$

form a set of intrinsic equations for a twisted ribbon or filament under the prescribed motion $\dot{\boldsymbol{X}} = q_T\boldsymbol{T} + q_N\boldsymbol{N} + q_B\boldsymbol{B}$. We note that equation (8) is of the same form as (6). This is because τ is in fact the twist of the Frenet ribbon (defined by $\boldsymbol{V} = \boldsymbol{N}$) as is verified by the equation $\tau = (\boldsymbol{N} \times (\mathrm{d}/\mathrm{d}s)\boldsymbol{N}) \cdot \boldsymbol{T}$. The Frenet ribbon is not a true physical ribbon as it is undefined at inflection points. Nevertheless away from inflection points it evolves locally as a ribbon with twist τ, and thus τ obeys an evolution equation of the form (6).

We digress for a moment to consider the connection between filament dynamics and integrable PDEs, a topic of perennial interest [3, 5]. In particular, we consider closed non-self-intersecting filaments with $\boldsymbol{X}$ and $\boldsymbol{V}$ periodic functions of σ. Under these circumstances we have the famous law $Lk = (2\pi)^{-1}Tw + Wr$ where Lk is the linking number of the curves $\boldsymbol{X}$ and $\boldsymbol{X} + \epsilon\boldsymbol{V}$ (an invariant unless the ribbon crosses itself), Tw is the total twist $\oint A\omega\,\mathrm{d}\sigma$, and Wr is the writhing number of $\boldsymbol{X}$ [10]. Formula (6) implies that under the dynamics of $\boldsymbol{X}$ we have $(\mathrm{d}/\mathrm{d}t)Tw = \oint \kappa(q_B' + A\tau q_N)\mathrm{d}\sigma$. As $(\mathrm{d}/\mathrm{d}t)Lk = 0$, this implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}Wr = -\frac{1}{2\pi}\oint \kappa(q_B' + A\tau q_N)\,\mathrm{d}\sigma.$$

Allowing for inflection points $\boldsymbol{X}(\sigma_1), \boldsymbol{X}(\sigma_2), \ldots, \boldsymbol{X}(\sigma_{n+1}) = \boldsymbol{X}(\sigma_1)$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}Wr = -\frac{1}{2\pi}\sum_{i=1}^{n}\int_{\sigma_i}^{\sigma_{i+1}} \kappa(q_B' + A\tau q_N)\,\mathrm{d}\sigma.$$

A mathematical derivation of this result has been previously obtained via a different method in [11].

Suppose we make the choice $\dot{\boldsymbol{X}} = \kappa\boldsymbol{B}$—this is the local induction approximation (LIA) of vortex filament dynamics [3]. Under these dynamics the function $\psi \equiv \kappa\exp(\mathrm{i}\int^{\sigma}\tau\,\mathrm{d}\sigma)$ satisfies the nonlinear Schrödinger equation (NLS) $\dot{\psi} + \psi'' + \frac{1}{2}\psi|\psi|^2 = 0$. We observe that

$$\frac{\mathrm{d}}{\mathrm{d}t}Wr = -\frac{1}{2\pi}\sum_{i=1}^{n}\int_{\sigma_i}^{\sigma_{i+1}} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\sigma}\kappa^2\,\mathrm{d}\sigma = 0.$$

Thus LIA conserves the writhing number up to crossings of the vortex filament. As a second example, choose $\dot{\boldsymbol{X}} = -\frac{1}{2}A\kappa^2\boldsymbol{T} - \kappa'\boldsymbol{N} - \kappa\tau\boldsymbol{B}$. In this case $\dot{A} = 0$ and if $A(\sigma, t=0) = 1$ then $A(\sigma, t) = 1$. Under these dynamics ψ satisfies the complex modified KdV equation $\dot{\psi} + \frac{3}{2}|\psi|^2\psi' + \psi''' = 0$ [5]. Here we have

$$\frac{\mathrm{d}}{\mathrm{d}t}Wr = \frac{1}{2\pi}\sum_{i=1}^{n}\int_{\sigma_i}^{\sigma_{i+1}} \frac{\mathrm{d}}{\mathrm{d}\sigma}(\kappa^2\tau)\,\mathrm{d}\sigma = 0. \tag{9}$$

Again we see conservation of writhing number under the curve dynamics up to filament crossing. (This may not be immediately obvious as in general $\tau = \infty$ at inflection points; however, using the formula

$$\tau = \frac{1}{\kappa^2}\left(\frac{\mathrm{d}}{\mathrm{d}s}\boldsymbol{X} \times \frac{\mathrm{d}^2}{\mathrm{d}s^2}\boldsymbol{X}\right)\cdot\frac{\mathrm{d}^3}{\mathrm{d}s^3}\boldsymbol{X}$$

(9) becomes clear.) More generally, Langer and Perline [5] have shown how to construct a sequence of arclength preserving velocities $\dot{\boldsymbol{X}}^{(j)}$, $0 \leqslant j < \infty$, such that under the action of $\dot{\boldsymbol{X}}^{(j)}$, ψ satisfies the jth equation in the NLS hierarchy (the first two velocities of this hierarchy are the two examples given above). The construction is carried out formally using the recursion formula

$$\begin{aligned}
\dot{\boldsymbol{X}}^{(0)} &= \kappa\boldsymbol{B}\\
\dot{\boldsymbol{X}}^{(j+1)} &= \boldsymbol{T}\times\dot{\boldsymbol{T}}^{(j)} + \mathcal{N}(\boldsymbol{T}\times\dot{\boldsymbol{T}}^{(j)})\boldsymbol{T}\\
&= \boldsymbol{\Omega}^{(j)} + \mathcal{N}(\boldsymbol{\Omega}^{(j)})\boldsymbol{T}
\end{aligned}$$

where the normalizer $\mathcal{N}(\boldsymbol{T}\times\dot{\boldsymbol{T}}^{(j)}) = \int^{s}\kappa((\mathrm{d}/\mathrm{d}s)q_B^{(j)} + \tau q_N^{(j)})\,\mathrm{d}s$ is the tangential velocity necessary to conserve arclength under the flow $\boldsymbol{T}\times\dot{\boldsymbol{T}}^{(j)}$ (see equation (7)). Thus the condition that $(\mathrm{d}/\mathrm{d}t)Wr = 0$ for the jth flow $\dot{\boldsymbol{X}}^{(j)}$ of the hierarchy is exactly the same condition necessary to ensure that the tangential component of the (j+1)th flow is well defined on closed curves, namely $\oint\kappa((\mathrm{d}/\mathrm{d}s)q_B^{(j)} + \tau q_N^{(j)})\mathrm{d}s = 0$. Thus in addition to the known connection of integrable curve dynamics to arclength preservation, we see that there is also a close connection to the geometric property of writhe conservation.

Using the derived kinematics we now introduce dynamical equations for a filament with twist under elastic forces beginning from the classical theory of thin filaments [6, 12]. The main new idea here is to follow the evolution of the twist rather than the evolution of the reference ribbon because, in principle, the twist is the more natural physical and computational object. Consider then an infinitesimal element of the filament from $\boldsymbol{X}(s)$ to $\boldsymbol{X} + \mathrm{d}\boldsymbol{X} = \boldsymbol{X}(s + \Delta s)$ of length Δs bounded by two cross sections. Let $\boldsymbol{F}(s,t)$ be the internal stress on the cross section at $\boldsymbol{X}(s)$. A force $\boldsymbol{F} + \Delta\boldsymbol{F}$ acts on the upper end of the element and a force $-\boldsymbol{F}$ acts on the lower end for a total force $\Delta\boldsymbol{F}$. Let $\boldsymbol{K}$ be the 'external' force on the filament per unit length which for our purposes we take to be the negative of the inertial force plus other forces $\boldsymbol{g}(s,t)$ (e.g. contact and drag forces). Then $(-\rho\ddot{\boldsymbol{X}} + \boldsymbol{g})\Delta s$ is the external force acting on the element where ρ is the density per unit length. We thus have $\Delta\boldsymbol{F} + (-\rho\ddot{\boldsymbol{X}} + \boldsymbol{g})\Delta s = 0$, so that

$$\rho\ddot{\boldsymbol{X}} = \frac{\mathrm{d}}{\mathrm{d}s}\boldsymbol{F} + \boldsymbol{g}. \tag{10}$$

A second equation comes from the balance of moments. Let $\boldsymbol{M}(s,t)$ be the moment of the internal stresses on a cross section. The total moment on the filament element (with respect to the upper end) is calculated as follows: the stresses on the upper end give a contribution $\boldsymbol{M} + \Delta\boldsymbol{M}$. The lower end gives a contribution $-\boldsymbol{M} + (-\Delta s\boldsymbol{T}) \times (-\boldsymbol{F})$. There is also an 'external' couple $\boldsymbol{J}$ which for our purposes we take to be the torsional inertia $-I\ddot{\theta}|_{\boldsymbol{X}}\Delta s$. Here I is the moment of inertia of the cross section and $\ddot{\theta}|_{\boldsymbol{X}}$ refers to $\ddot{\theta}$ with the filament position $\boldsymbol{X}$ held fixed. Hence

$$\frac{\mathrm{d}}{\mathrm{d}s}\boldsymbol{M} = \boldsymbol{F} \times \boldsymbol{T} + I\ddot{\theta}|_{\boldsymbol{X}}\boldsymbol{T}. \tag{11}$$

Elasticity theory gives us $\boldsymbol{M}$ as a function of $\boldsymbol{X}$ (using the thin filament assumption). In the case of linear elasticity,

$$\boldsymbol{M} = EI\kappa\boldsymbol{B} + C\omega\boldsymbol{T} \tag{12}$$

[6] where E is the Young's modulus and C the torsional rigidity.

We now write down dynamical equations for the linearly elastic filament. First, the tangential component of (11) gives $C(\mathrm{d}/\mathrm{d}s)\omega = I\ddot{\theta}|_{\boldsymbol{X}}$. Using (3) we can write this as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\mathrm{d}}{\mathrm{d}t}(A\omega) - A\kappa\left(\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{T}\right)_{\boldsymbol{B}}\right] = \frac{C}{I}\frac{\mathrm{d}}{\mathrm{d}\sigma}A\frac{\mathrm{d}}{\mathrm{d}\sigma}(A\omega). \tag{13}$$

This is the traditional torsional wave equation of thin rod theory plus a term arising from the motion of the filament itself.

Now, using (12) and (4), the normal and binormal components of (11) are

$$F_N = -EI\frac{\mathrm{d}}{\mathrm{d}s}\kappa$$
$$F_B = C\omega\kappa - EI\kappa\tau.$$

The tension F_T is undetermined from (11) and must be considered separately. F_T is the longitudinal stress at the cross section times the area $\pi\epsilon^2$ of a cross section and is proportional to the local filament expansion. Thus for instance if $\boldsymbol{X}(\sigma)$ is a material parametrization of the filament such that $\mathrm{d}\sigma/\mathrm{d}s = 1$ in a relaxed state (no tension), then in a general state $F_T = \pi\epsilon^2 E(A-1)$.

Finally, using (10) we obtain

$$\begin{aligned}\rho\ddot{\boldsymbol{X}} = {} & \left(\frac{\mathrm{d}}{\mathrm{d}s}F_T - \frac{1}{2}EI\frac{\mathrm{d}}{\mathrm{d}s}\kappa^2 + g_T\right)\boldsymbol{T} + \left(-EI\frac{d^2}{\mathrm{d}s^2}\kappa + \kappa F_T - C\kappa\tau\omega + EI\tau^2\kappa + g_N\right)\boldsymbol{N} \\ & + \left(\frac{\mathrm{d}}{\mathrm{d}s}(C\kappa\omega - EI\kappa\tau) - EI\tau\frac{\mathrm{d}}{\mathrm{d}s}\kappa + g_B\right)\boldsymbol{B}\end{aligned} \tag{14}$$

(13) and (14) are the evolution equations for a thin isotropic filament under linear elastic forces.

Acknowledgments

The authors would like to thank R Hamilton and J Aldinger for valuable discussions and J B Keller for a helpful conversation. In addition, the authors thank one of the referees for bringing reference [12] to their attention. This work was supported by DOE grant DE-FG03-93-ER25174 and an NSF post-doctoral fellowship.

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